Bachelorarbeit

On the Number of Arrangements of Pseudolines

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Abstract

Let B_n be the number of non-isomorphic marked arrangements of n pseudolines. We demonstrate that $B_n \geq 2^{cn^2 - O(n\log(n))}$ for some $c > 0.2144$, exceeding the previous best lower bound $c > 0.2083$ by Dumitrescu and Mandal (2020). The problem of estimating B_n was first posed by Goodman and Pollak in 1983. They established the lower bound $B_n \geq 2^{\frac{n^2}{8}}$. Knuth improved the lower bound to $B_n \geq 2^{\frac{n^2}{6} - O(n)}$ and found the upper bound $B_n \leq 3^{\binom{n}{2}} \approx 2^{0.792n^2}$. The upper bound has also since been improved upon, most recently by Felsner and Valtr in 2011. They discovered that $B_n \leq 2^{0.657n^2}$ for sufficiently large n.

The argument used in this thesis is based on gluing together partial arrangements that are consistent with some outside behavior, which we make precise using boundary bipermutations. For each of these "patches" we then calculate the number of partial arrangements consistent with that outside behavior using an algorithm that is also provided. From this the bound follows using known techniques.

Acknowledgements

First and foremost, I would like to thank my thesis supervisor Prof. Dr. Stefan Felsner for supporting me with helpful discussions and providing me with a very interesting topic. Furthermore, I thank Dr. Manfred Scheucher for giving many helpful suggestions for the algorithm and teaching me how to use the compute cluster. Finally, I thank my family for the support they gave me over the years and especially during the past weeks.

Contents

1 Introduction

Arrangements of lines and hyperplanes are simple and natural objects both in projective and euclidean space and have been studied for hundreds of years in discrete and computational geometry. In [8] Levi generalized arrangements of lines by introducing arrangements of pseudolines, which behave like lines topologically, but not necessarily geometrically. A good exposition on arrangements of lines and pseudolines was given by Grünbaum in [6]. In this thesis we will focus on arrangements of pseudolines in euclidean space.

An *arrangement of pseudolines* in $\mathbb{R}^2$ is a finite family of simple curves, each of which approaches infinity in both directions and any two of which intersect in exactly one point, where they cross. It is called *simple* if no three pseudolines intersect in a point. A *marked arrangement* is an arrangement together with a distinguished unbounded *north-cell*. Two marked arrangements are *isomorphic* if one can be mapped to the other by an orientation preserving homeomorphism of the plane that also preserves the north-cell. (See Definitions 1 and 2 for more detail.)

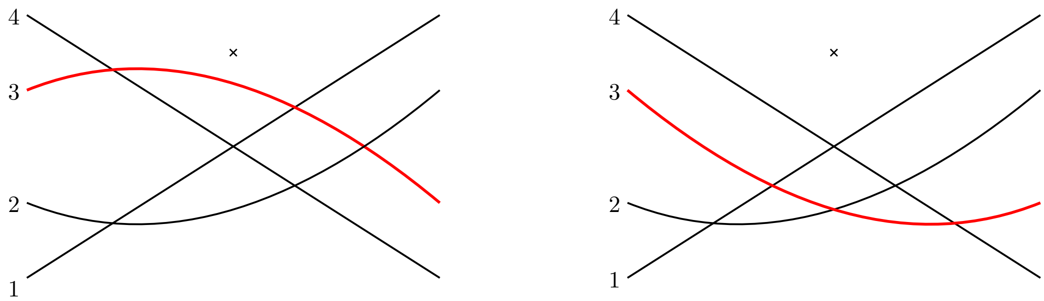

Figure 1: Two non-isomorphic marked arrangements. The north-cell is distinguished by a cross.

An arrangement is called *stretchable* if it is isomorphic to an arrangement of straight lines. A natural question to ask is whether every arrangement of pseudolines is stretchable. To see why this is not the case, consider the arrangement of lines in Figure 2. The points X,Y and Z must lie on a line f by Pappus's theorem. By perturbing the central line f around Y, we create an arrangement of pseudolines that cannot be isomorphic to any arrangement of lines. This argument comes from [8]. We get a simple arrangement with the same property by perturbing f in the other direction around X,Z and shifting a few of the lines in a way which only moves the intersection points X,Z and Y in the direction of the perturbation of f, see Figure 2. In fact, there are many non-stretchable arrangements. So many, that the number B_n of non-isomorphic arrangements of n pseudolines grows as $2^{\Theta(n^2)}$ (a lower bound is given in [5, Proposition 6.2], an upper bound can be derived from [2, Theorem 2.7]), while the number of non-isomorphic arrangements of n lines only grows as $2^{O(n \log n)}$ [10, Chapter 6.2]. In particular, the number of arrangements of lines represents a fraction of the total number of arrangements that tends to zero as n tends towards infinity.

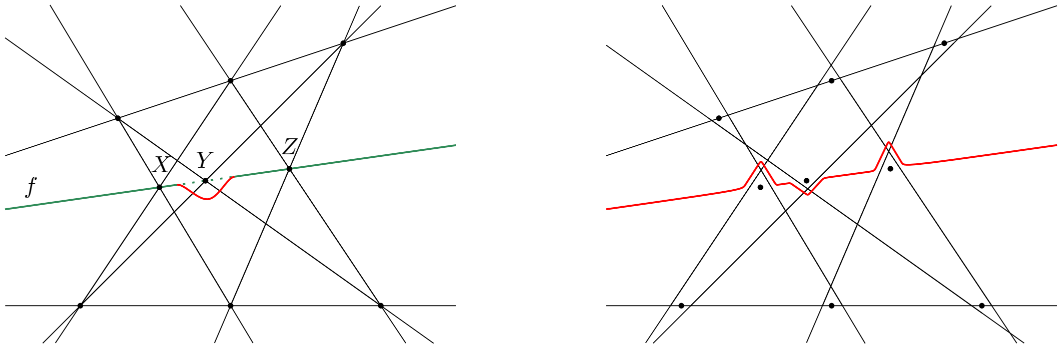

Figure 2: Non-stretchable arrangements. Left: An arrangement resulting from the perturbation of the central line in a Pappus arrangement. Right: A simple version of the same idea.

Our goal in this thesis is to learn more about the growth rate of B_n. In particular, we want to bound the multiplicative factor of the leading term of $b_n := \log_2 B_n = \Theta(n^2)$. That is the number

$$c_{\sup} := \sup\{c \in \mathbb{R} \mid b_n \geq cn^2 \text{ for all sufficiently large } n \in \mathbb{N}\}.$$

A lot of work has been done on this question already, starting with Goodman and Pollak in [5], where they established the lower bound $c_{\sup} \geq \frac{1}{8}$. In [7, Section 9] Knuth proved the upper bound $B_n \leq 3^{\binom{n+1}{2}}$ giving us $c_{\sup} \leq \frac{1}{2}\log_2(3) \approx 0.79$. This was achieved by bounding the number γ_n of ways a new pseudoline ℓ can be inserted into an arbitrary existing arrangement $\mathcal{A}$ of n pseudolines, such that ℓ starts in the north cell, ends in the south-cell and introduces no multi-crossings. Each such possibility is called a *cutpath*. Since every arrangement of $n+1$ pseudolines can be constructed in this way we have $B_{n+1} \leq \gamma_n B_n$. Knuth showed $\gamma_n \leq 3^n$, from which the bound can be gotten through induction. The bound on γ_n has since been sharpened. In particular in [4], where $\gamma_n \leq 4n \cdot 2.486976^n$ was shown, yielding $c_{\sup} \leq 0.6571$, the current best upper bound. The current best lower bound $c_{\sup} > 0.2083$ was established by Dumitrescu and Mandal in [1].

In this thesis we establish the new lower bound $c_{\sup} > 0.2144$. The approach used is in the spirit of those employed in [1] or [4, Section 4]. One starts with a partial arrangement of n straight lines, that are separated into k strips of parallel lines, see for example Figure 4. Now one constructs arrangements of pseudolines, by considering the interaction within and between strips separately. First one bounds from below the number $F_k(n)$ of ways the pseudolines from different strips can intersect while remaining consistent with the global behavior of the lines. Then this bound is applied recursively to the lines within each strip, see Lemma 2.

What sets our approach apart is the method we use to lower bound $F_k(n)$. It is based on separating the region of intersection of the strips into sub-regions (or “patches”) of constant size. For each patch we calculate the exact number of ways the pseudolines can interact within the patch without changing the behavior at its boundary. This is done computationally using the algorithm we provided. We then multiply the results to get a lower bound on $F_k(n)$.

All of this is fleshed out in Section 2.3, where we describe in detail the strategy that was used to get our bound, as well as some older bounds. Before that, in Section 2.1, we define the objects that will be used. In particular arrangements and partial arrangements of pseudolines as well as boundary bipermutations will be defined there. We then use Section 2.2 to show a few basic properties of partial arrangements. In Section 3 we review the most common ways to represent arrangements combinatorially and look at previous approaches to bounding c_{sup}. This section is not necessary to understand the proof of the main theorem, but could be helpful to place our approach in relation to previous work in the subject. Finally, in Section 4.1 we prove the lower bound using the methods outlined above. In the proof we use constants from Tables 1 2 that were determined using the algorithm explained in Section 4.2.

2 Preliminaries

We begin the main body of this thesis with Section 2.1 by providing the definitions of the objects we will use, including arrangement, partial arrangement and boundary bipermutation. Then, in Section 2.2, we prove some results about partial arrangements which are fundamental for the proof of the main theorem, Theorem 3. The main result needed for the new bound is Corollary 2.3, which ensures that all of the arrangements we will construct are non-isomorphic. Theorem 1 is not needed for the new bound, but may be of interest in itself. Finally, in Section 2.3 we explain how to obtain non-isomorphic arrangements from particular partial arrangements of straight lines, ending on the important Lemma 2, which will also play a central role for the new bound.

2.1 Basic Definitions

Even though we have had a rough explanation of arrangements of pseudolines in the introduction, a more precise definition is needed and provided here. Usually it is not useful to define a pseudoline by itself, as it is the interaction between pseudolines within an arrangement that is of interest. In this case it makes sense, since pseudolines will feature in the definitions of both arrangements and partial arrangements of pseudolines.

Definition 1. *A* **pseudoline** *in $\mathbb{R}^2$ is a simple curve approaching infinity in both directions. More precisely it is an equivalence class of continuous injections $\gamma : \mathbb{R} \to \mathbb{R}^2$ that satisfy $\lim_{t\to\pm\infty} \|\gamma(t)\| = \infty$, where two functions γ_1, γ_2 are considered equivalent if and only if there exists a homeomorphism $\tau : \mathbb{R} \to \mathbb{R}$ such that $\gamma_1 = \gamma_2 \circ \tau$. (Similarly one can define the* *notion of an* **oriented pseudoline** *by requiring the homeomorphisms τ to be increasing.) To avoid pathological cases we will also require pseudolines to coincide with rays (one in each direction) outside of some bounded region.*

Definition 2. *A finite family of pseudolines is called an* **arrangement** *if each pair of pseudolines intersects in exactly one point, where the pseudolines cross. It is called* **simple** *if no three lines intersect in a point.*

An arrangement $\mathcal{A}$ separates the plane into *vertices* (intersection points), *edges* (pseudoline segments) and *cells* (connected components of $\mathbb{R}^2 \setminus \bigcup \mathcal{A}$). A *marked arrangement* is an arrangement with a distinguished unbounded cell called *north-cell*. In a marked arrangement, there exists exactly one *south-cell*, which is separated from the north-cell by every pseudoline in the arrangement. We obtain a natural orientation on the pseudolines by demanding that the north-cell be to the left and the south-cell to the right of each oriented pseudoline. Two marked arrangements are considered *isomorphic* if they can be transformed into each other by a homeomorphism of the plane, which preserves both the north-cell and the orientations of the pseudolines.

From now on, "arrangement" will always refer to an equivalence class of marked arrangements, unless specified otherwise. If the north cell is not specified, assume it is the cell that is unbounded in the positive y-direction from $(0,0)$.

Definition 3. *A* **partial arrangement of pseudolines** *is a finite family of pseudolines, any two of which may intersect and cross once or not intersect at all. Simpleness is defined the same way it is with arrangements.*

Let $\mathcal{A} = \{\ell_1, \dots, \ell_n\}$ be a partial arrangement of n pseudolines. There are only finitely many crossings and bounded edges in $\mathcal{A}$. Let $U \subset \mathbb{R}^2$ be some open, bounded, convex region containing all of them and containing exactly one section of each pseudoline. A disc centered at the origin with sufficiently large radius always satisfies these conditions, because there are only finitely many pseudolines and we required each to coincide with rays outside of some bounded region. We can distinguish one point $p \in \partial U \setminus \bigcup_{i \in [n]} \ell_i$ on the boundary of U and call it the *north-pole*, which gives rise to the notion of a *marked partial arrangement* $(\mathcal{A}, U, p)$.

Let C be the unbounded cell containing p. We will call C the north-cell of the marked partial arrangement. Note however, that there may be no south-cell that is separated from the north cell by every pseudoline. In fact, it may even be the case, that $\mathbb{R}^2 \setminus C$ is disconnected, as the partial arrangement of two parallel lines shows. Nonetheless we can define a natural orientation on the pseudolines by demanding that the north-cell is to the left of every one of them. We will call two marked partial arrangements $(\mathcal{A}, U, p)$, $(\mathcal{A}', U', p')$ *isomorphic*, if one can be transformed into the other by a homeomorphism of the plane that preserves the natural orientations of the pseudolines and maps U to U' and p to p'. If $\mathbb{R}^2 \setminus C$ is connected, we may define a marked partial arrangement by specifying the north-cell instead of the north-pole and U.

Starting at p we can walk around the boundary of U (in clockwise direction) and record the order in which the pseudolines cross it. The resulting object will be a map $\sigma : [2n] \to [n]$, which attains every value in $[n]$ exactly twice, a *bipermutation*.

Definition 4. *We will call σ the* **boundary bipermutation** *(or just bipermutation) of $\mathcal{A}$. Let $S_n^{(2)} := \{\sigma : [2n] \to [n] \,|\, \sigma \text{ is a bipermutation}\}$ denote the set of all bipermutations on n elements. Partial arrangements with a given boundary bipermutation σ will be called* **consistent** *with σ.*

For any $\sigma \in S_n^{(2)}$ let B_σ denote the number of non-isomorphic partial arrangements that are consistent with σ. There are a few notable operations on bipermutations that leave B_σ unchanged. A *cyclic shift* on a bipermutation $\sigma = (a_1, \ldots, a_n)$ maps it to a bipermutation of the form $\sigma' = (a_k, \ldots, a_n, a_1, \ldots, a_{k-1})$. It corresponds to choosing a different north-pole in a partial arrangement. Very similarly, a $reflection$ maps σ to $(a_n, a_{n-1}, \ldots, a_1)$ and corresponds to a reflection of the plane. Finally, a *relabeling* of σ will have the form $(\pi(a_1), \ldots, \pi(a_n))$, where $\pi \in S_n$ is a permutation of the n labels of the pseudolines.

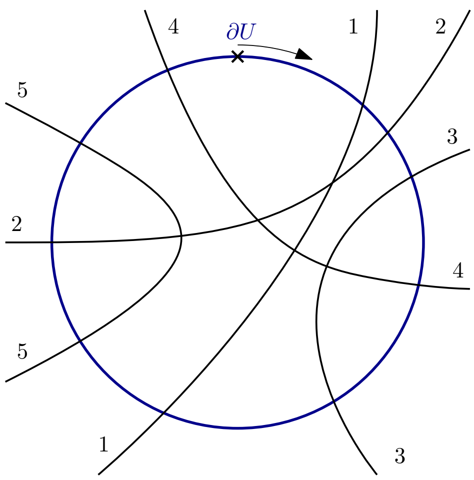

Figure 3: A partial arrangement with boundary bipermutation (1,2,3,4,3,1,5,2,5,4). Note that it is not of the form (1,2,3,. . . n,. . .). The north-cell is marked by a cross. In this partial arrangement there exists no cell that is separated from the north-cell by every pseudoline.

Definition 5. *Let $\mathcal{A}$ be a partial arrangement. Given an open, bounded, convex region U that contains at most one section of each pseudoline and such that ∂U contains no crossings as well as a point $p \in \partial U$ we can define the marked partial arrangement $(\mathcal{B}, U, p)$* **induced by $\mathcal{A}$ and** *U as follows.*

The pseudolines in $\mathcal{B}$ correspond to those pseudolines in $\mathcal{A}$ that intersect U. $\mathcal{B}$ coincides with $\mathcal{A}$ on U and there are no crossings of pseudolines of $\mathcal{B}$ outside of U. This can be achieved by extending the restricted pseudolines from ∂U as rays from a common point in U. We can now choose p as the north-pole of the marked partial arrangement.

Note that, while $\mathcal{A}$ has to be one specific partial arrangement, the induced partial arrangement is only unique up to isomorphism outside of U. We will however still write *the* induced partial arrangement, since the behavior of the pseudolines outside of U is not interesting to us.

2.2 Some Properties of Partial Arrangements

In this section we show some basic results about partial arrangements. In particular, we show that any bipermutation is realized by at least one partial arrangement and that any partial arrangement can in some sense be extended into a full arrangement of pseudolines. Finally we show that a change done to an arrangement in one region cannot be undone outside of that region, which is needed for our proof of the new bound, Theorem 3.

Proposition 1. *For any bipermutation $\sigma \in S_n^{(2)}$ there exists at least one consistent simple partial arrangement. In particular $B_\sigma \geq 1$.*

Proof. Consider a disc U centered at the origin, with $2n$ equally spaced points around its boundary. Starting at the top, we can label the points according to σ. We will connect two points with the same label using a straight line. We then continue the lines using rays pointing away from the origin. This yields a potentially non-simple arrangement of pseudolines. It can be simplified by resolving multi-crossings locally. $\square$

Proposition 2. *The boundary bipermutation of a partial arrangement completely determines whether two pseudolines in the partial arrangement cross.*

Proof. Let $\sigma \in S_n^{(2)}$ be a bipermutation. Let $(\mathcal{A}, U, p)$ be a partial arrangement consistent with σ and let ℓ_a, ℓ_b be two pseudolines in $\mathcal{A}$. By the Jordan Curve Theorem the pseudoline ℓ_a separates U into two disconnected halves. If a and b appear in σ in a cross-configuration (i.e. *abab* or *baba*) then the intersections of ℓ_b and ∂U lie on different halves of U and the pseudolines must cross. If they appear in a parallel configuration (i.e. *abba*, *aabb*,…) then the pseudolines cannot cross, or they would have to cross twice. $\square$

We will say that two elements $a, b \in [n]$ cross in $\sigma \in S_n^{(2)}$ if they appear in a cross-configuration (i.e. *abab* or *baba*). That is if, in consistent partial arrangements, the pseudolines corresponding to a and b cross.

Lemma 1. *Let $\sigma \in S_n^{(2)}$ be some bipermutation that has at least one pair of non-crossing elements. Then σ also has a pair of neighboring elements that do not cross. We call $c, d \in [n]$ neighboring if there exists an index $i \in [2n-1]$ with $\sigma(i) = c$, $\sigma(i+1) = d$ or if $\sigma(2n) = c$, $\sigma(1) = d$.*

Proof. Let $a, b \in [n]$ be two non-crossing elements in σ. We can assume σ to be of the form $(a, \ldots b, \ldots b, \ldots a, \ldots)$ otherwise do a cyclic shift, which will not change whether elements cross or neighbor each other. Let i be the last index in the stretch $(a, \ldots b)$, excluding b, such that $c := \sigma(i)$ does not cross b. In particular c's direct neighbor $d := \sigma(i+1)$ either crosses b or is itself b. In the second case c and b are non-crossing neighbors and we are done. Otherwise we can shift the bipermutation to reach the form $(c, \ldots c, d, \ldots b, \ldots b, \ldots)$. Since d crosses b, its second appearance has to be between the two appearances of b. Therefore it cannot cross c. □

Theorem 1. *Any partial arrangement can be transformed into a full arrangement by repeatedly crossing unbounded edges.*

Proof. Let $\mathcal{A}$ be a partial arrangement of n pseudolines and let σ be its boundary bipermutation. If $\mathcal{A}$ is an arrangement, we are done. Otherwise Proposition 2 ensures that there are two neighboring pseudolines that do not cross. We can create a new partial arrangement $\mathcal{A}'$ by crossing those pseudolines in the unbounded cell, where they neighbor each other. Since $\mathcal{A}$ was chosen arbitrarily and $\mathcal{A}'$ has strictly more crossings than $\mathcal{A}$, we can do this repeatedly until there are no pairs of non-crossing pseudolines left and we reach an arrangement. □

Corollary 1.1. *Let $\mathcal{A}$ be a partial arrangement of n pseudolines and let U be a disc containing all of the crossings in $\mathcal{A}$. Then there exists a full arrangement of n pseudolines, that coincides with $\mathcal{A}$ on U.*

The strategy used in the proof of the new bound can be described as stitching together partial arrangements to get full arrangements. We want these full arrangements to be non-isomorphic if we start with non-isomorphic patches. For this to work we need to ensure that a change in one of these patches cannot be undone somewhere else. This is expressed in the following theorem.

Theorem 2. *Let $\mathcal{A} = \{\ell_1, \ell_2, \ldots \ell_n\}$, $\mathcal{A}' = \{\ell'_1, \ell'_2, \ldots \ell'_n\}$, $n \in \mathbb{N}$ be two isomorphic marked arrangements with isomorphism $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$, such that $\Phi(\ell_i) = \ell'_i$ for all $i \in [n]$.*

Let U be an open, bounded, convex region that contains at most one section of each pseudoline (from both $\mathcal{A}$ and $\mathcal{A}'$) such that ∂U does not contain intersections of pseudolines and such that $\mathcal{A}$ and $\mathcal{A}'$ agree on ∂U, meaning $\ell_i \cap \partial U = \ell'_i \cap \partial U$ for all $i \in [n]$. Let $p \in \partial U$ be an arbitrary point on the boundary of U.

Let $\mathcal{B}$, $\mathcal{B}'$ be the partial arrangements that are induced by $\mathcal{A}$ and $\mathcal{A}'$ on U. Then $\mathcal{B}$ and $\mathcal{B}'$ are isomorphic.

Proof. We need to construct a homeomorphism $\widetilde{\Phi} : \mathbb{R}^2 \to \mathbb{R}^2$ that preserves U and p and maps the pseudolines in $\mathcal{B}$ to those in $\mathcal{B}'$. We will do this by showing, that there exists a homeomorphism Ψ that transforms $\Phi(U)$ into U and $\Phi(p)$ into p while preserving all of the pseudolines in $\mathcal{B}'$ (as well as their orientations). Then $\Phi \circ \Psi$ is an isomorphism between $\mathcal{B}$ and $\mathcal{B}'$.

Since $\partial\Phi(U)$ intersects the same pseudolines in $\mathcal{A}'$ in the same order when starting from $\Phi(p)$ as ∂U does starting from p, we know that $\Phi(U)$ and U contain sections as well as crossings of the same pseudolines. In particular, if $\Phi(U) \cap U = \emptyset$ then U contains no intersections and the statement of the theorem is trivially true. Otherwise we have that all crossings and a section of each pseudoline in $\mathcal{B}'$ lie in $\Phi(U) \cap U$. Hence the symmetric difference $\Phi(U) \setminus U \cup U \setminus \Phi(U)$ only contains segments of pseudolines that run directly between $\partial\Phi(U)$ and ∂U without crossing other pseudolines. Therefore there is nothing within the symmetric difference preventing us from continuously transforming from $\Phi(U)$ to U and back without effecting the pseudolines.

On the other hand, there are only finitely many crossings of pseudolines from $\mathcal{B}'$, none of which lie on $\partial\Phi(U) \cup \partial U$. Since both the set of crossings and the boundaries are compact, there is some distance $d > 0$ between the two sets. Further, because there are only finitely many pseudolines, there is some $0 < \varepsilon \leq d$ such that $\bigcup_{x \in \partial\Phi(U) \cup \partial U} U_\varepsilon(x)$ only contains pseudoline segments, that are connected to $\partial\Phi(U) \cup \partial U$. (If this were not the case, the closure of those sections would again be a closed set that is disjoint from $\partial\Phi(U) \cup \partial U$ and there would be a positive distance.) Therefore we can continuously transform between $\Phi(U)$ to U without obstruction. $\square$

Corollary 2.1. *With the same requirements as above (except for $\mathcal{A}$, $\mathcal{A}'$ being isomorphic), if $\mathcal{B}$, $\mathcal{B}'$ are non-isomorphic, then $\mathcal{A}$, $\mathcal{A}'$ also cannot be isomorphic.*

Corollary 2.2. *For any bipermutation $\sigma \in S_n^{(2)}$ there are at most B_n partial arrangements consistent with σ. In other words we have $B_\sigma \leq B_n$.*

Corollary 2.3. *Let $\mathcal{A}$ be a marked arrangement and let $U_1, \ldots U_n$, $n \in \mathbb{N}$ be a number of disjoint, open, bounded, convex regions such that each contains at most one section of each pseudoline and whose boundaries contain no crossings of pseudolines and touch no pseudolines. For all $i \in [n]$ let $p_i \in \partial U_i$ be some point on the boundary of U_i and let $\mathcal{B}_i$, σ_i be the partial arrangements and their bipermutations induced by $\mathcal{A}$ on U_i.*

Exchanging any collection of the $\mathcal{B}_i$ with a non-isomorphic partial arrangements with the same bipermutation leads to an arrangement $\mathcal{A}'$ that in not isomorphic to $\mathcal{A}$. In particular,

we have
$$\prod_{i\in[n]} B_{\sigma_i} \leq B_n.$$

2.3 Arrangements Based on Partial Arrangements of Straight Lines

We end Section 2 by turning our attention to a general strategy used to construct and count non-isomorphic arrangements. Later we will use it to prove a new lower bound on c_{sup}. The same strategy has been employed before in [1] and [4, Section 4]. The new aspects of our approach will be covered in Section 4.

Let $k \in \mathbb{N}$, $k \geq 3$ be fixed once and for all. For any $n \in \mathbb{N}$ we start with a partial arrangement of n straight lines, most of which are organized into k strips of m equidistant parallel lines each, such that $n = km + r$ for some r that satisfies $0 \leq r < O(1)$ when viewed as a function of n. For convenience we will suppress the dependence of both r and m on n. The r unused lines will not contribute to the bound and can be safely inserted into the arrangement at the end. Given the natural labeling of the lines shown in Figure 4 the partial arrangement will have a boundary bipermutation σ of the form

$$\sigma = (1,2,3,\dots n, m, m-1,\dots 1, 2m, 2m-1,\dots m+1,\dots km,\dots (k-1)m+1).$$

Next we bound from below the number $F_k(n)$ of simple partial arrangements that are consistent with σ. Finally, we need to consider the m pseudolines corresponding to the lines within each strip. To get an arrangement, these pseudolines will have to cross as well. We will assume that those crossings will happen independently outside of some region U containing all of the crossings of lines from different strips, see Figure 4. The number of arrangements $T(n) := B_n$ then satisfies
$$T(n) \geq F_k(n)T(m)^k. \tag{1}$$

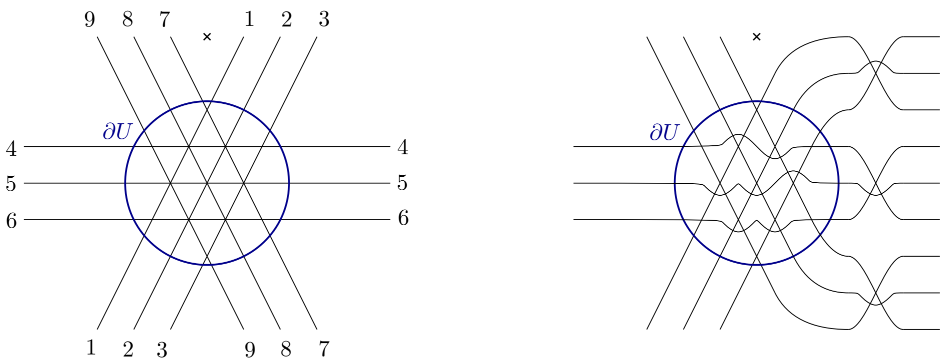

Figure 4: Left: A partial arrangement of $k \cdot m = 9$ straight lines organized into $k = 3$ strips of $m = 3$ lines each. Right: One possible arrangement arising from the construction described above. The crosses mark the respective north-cells.

Recursively applying the bound on $F_k(n)$ to the strips yields an improved bound on $T(n)$. This contribution is quantified by the following lemma. It seems to have been used implicitly in proofs of previous lower bounds.

Lemma 2. *Using the terms defined above, if $F_k(n) \geq 2^{c'n^2-O(n)}$, for some $c' > 0$ then $T(n) \geq 2^{\frac{k}{k-1}c'n^2-O(n\log n)}$. In particular $c_{\sup} \geq \frac{k}{k-1}c'$.*

Proof. Let $c := \frac{k}{k-1}c'$ and let $L \geq 0$ be a constant such that $F_k(n) \geq 2^{c'n^2-Ln}$ for all $n \geq 1$. Further let $R \geq 0$ be a constant, with which $n - R \leq km \leq n$ is satisfied for all $n \geq 1$. Define $\tilde{L} := L + 2\frac{c}{k}R$ and $G(n) := \tilde{L}n\log_k(\tilde{L}n)$ for all $n \geq 1$. We show by induction, that for all $n \geq 1$

$$T(n) \geq 2^{cn^2-G(n)}. \tag{2}$$

As a base case consider $n = 1, \ldots k$. We can assume inequality (2) to be satisfied, otherwise we increase L. For the induction step we get

$$\begin{aligned}
\log_2 T(n) &\overset{(1)}{\geq} \log_2\left(F_k(n)T(m)^k\right) \\
&\geq c'n^2 - Ln + k\left(cm^2 - G(m)\right) \\
&= c'n^2 - Ln + \frac{c}{k}(km)^2 - kG(m) \\
&\geq c'n^2 - Ln + \frac{c}{k}(n-R)^2 - kG(m) \\
&= c'n^2 + \frac{c}{k}n^2 - Ln - 2\frac{c}{k}Rn - kG(m) + \frac{c}{k}R^2,
\end{aligned}$$

where we used the induction hypothesis in the second step. Splitting this sum up, we have

$$c'n^2 + \frac{c}{k}n^2 = \left(\frac{k-1}{k} + \frac{1}{k}\right)cn^2 = cn^2.$$

Furthermore we have

$$\begin{aligned}
Ln + 2\frac{c}{k}Rn + kG\left(\left\lfloor\frac{n}{k}\right\rfloor\right) &= \tilde{L}n + k\tilde{L}\left\lfloor\frac{n}{k}\right\rfloor\log_k\left(\tilde{L}\left\lfloor\frac{n}{k}\right\rfloor\right) \\
&\leq \tilde{L}n + \tilde{L}n\log_k\left(\tilde{L}\left\lfloor\frac{n}{k}\right\rfloor\right) \\
&= \tilde{L}n\left(1 + \log_k\left(\tilde{L}\left\lfloor\frac{n}{k}\right\rfloor\right)\right) \\
&= \tilde{L}n\log_k\left(\tilde{L}k\left\lfloor\frac{n}{k}\right\rfloor\right) \\
&\leq \tilde{L}n\log_k(\tilde{L}n) \\
&= G(n).
\end{aligned}$$

And lastly, because $c, k > 0$, we have

$$\frac{c}{k} R^2 \geq 0.$$

Taking all of this into account, it follows that $\log_2 T(n) \geq cn^2 - G(n)$.
Since $G(n) \in O(n \log n)$ this proves the lemma. $\square$

3 Background

There are many ways to encode arrangements of pseudolines. In this section we review three of them and look at how some of them were applied to get bounds on c_{sup}. This section is not necessary to understand the proof of the new lower bound. Instead it serves as background information that may help to evaluate the method used in this thesis and compare it to other related works.

3.1 Combinatorial Representations of Arrangements

Local sequences. Let $\mathcal{A}$ be a simple arrangement of n pseudolines. To each pseudoline ℓ_i in $\mathcal{A}$ associate its *local sequence*, a permutation a_i of $[n] \setminus \{i\}$ that records the order in which ℓ_i crosses the other pseudolines. The family $(a_i)_{i \in [n]}$ is called the family of local sequences of $\mathcal{A}$ and is both uniquely determined by $\mathcal{A}$ and uniquely determines $\mathcal{A}$ [3, Theorem 6.6]. However, not every family of local sequences corresponds to an arrangement, as the example $a_1 = (2,3,4)$, $a_2 = (1,4,3)$, $a_3 = (1,*,*)$, $a_4 = (1,2,3)$ shows.

Wiring diagrams. Three closely related representations of simple arrangements are *wiring diagrams*, *allowable sequences* and *reflection networks*. Reflection networks are sequences of adjacent transpositions that transform tuples $(x_1, \ldots x_n)$ into their reflections $(x_n, \ldots x_1)$ [7, Section 8]. Composing the first i of those transpositions yields a sequence of permutations of $[n]$, the first one being the identity and the last one being the reverse permutation $(n, n-1, \ldots 1)$. Such a sequence is called an allowable sequence if each pair of elements $a, b \in [n]$ reverses their order exactly once [3, Section 6.2].

A wiring diagram is a standardized drawing of an arrangement, from which the corresponding allowable sequence and reflection network be read off easily. In it pseudolines are restricted to a set of horizontal lines (wires) except for the regions where they cross. Each crossing should happen in a distinct vertical strip, see Figure 5. We can get a wiring diagram from any simple marked arrangement, by *sweeping* it [3, Chapter 6.1 and 6.2]. Start with a curve α_1 with endpoints p_1 in the south-face and p_2 in the north-face, that only passes through unbounded faces or edges. Construct the next curve α_{i+1} by passing α_i over exactly one new vertex of the arrangement and repeat until all vertices have been passed over. Each curve α_i crosses the pseudolines in some order, which corresponds to the i-th permutation of the related allowable sequence, assuming the pseudolines to be labeled in such a way, that α_1 crosses them in ascending order. The orders of two consecutive curves α_i, α_{i+1} differ by an adjacent transposition. We can now draw the arrangement as a wiring diagram by crossing the pairs of lines corresponding to these transpositions in the $x \in [i, i+1)$ strip. We get the related reflection network by recording the transpositions in the order in which they appear. Note that none of these representations are uniquely determined by the arrangement, as shown in Figure 5.

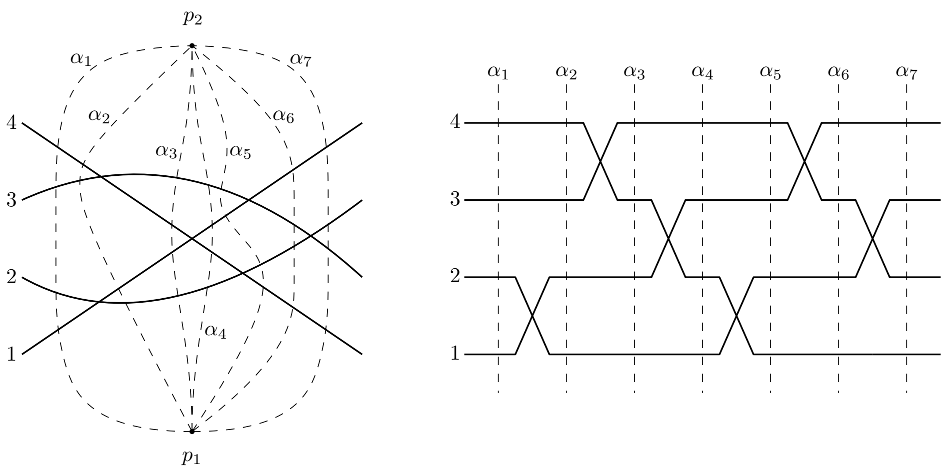

Figure 5: Left: A sweep of an arrangement. Right: The corresponding wiring-diagram. The reflection network $[1,2][3,4][2,3][1,2][3,4][2,3]$ can be read off of the wiring-diagram by looking at the wire-crossings. Note that for example the first two crossings could be swapped, yielding an equivalent but different reflection network.

Zonotopal duals. This last representation is particularly nice. The *zonotopal dual* of an arrangement is a particular zonotopal tiling. To explain these notions we first review some basic concepts. A *zonotope* in the plane $\mathbb{R}^2$ is defined as the Minkowski sum of a set of n line segments. It is easy to see that it is a centrally symmetric $2n$-gon, where opposing sides correspond to one of the line segments. By a translation we will assume the center to be at the origin. Consequently we arrive at the following form $Z(V)$. For a set of vectors $V = \{v_1, \ldots v_n\} \subset \mathbb{R}^2$ define the zonotope

$$Z(V) := \{\sum_{i=1}^{n} c_i v_i \,|\, c_i \in [-1,1]\} = [-v_1, v_1] + \ldots + [-v_n, v_n],$$

where $+$ denotes the Minkowski sum of the line segments $[-v_i, v_i]$. A *zonotopal tiling* of $Z(V)$ is a covering of $Z(V)$ using translates (but no rotations) of zonotopes $Z(V_i)$, where $V_i \subset V$, such that any two distinct zonotopes of the tiling intersect in a common edge or vertex. We remark that a zonotopal tiling defines a drawing of a graph, with zonotopes as faces.

The *zonotopal dual* of a (possibly non-simple) arrangement is a zonotopal tiling whose underlying graph is the dual graph of the arrangement, see Figure 5. We remark that the boundary of the unbounded face of the dual graph is formed by the edges of the zonotope $Z(V)$ and that the bounded faces form the zonotopal tiling of $Z(V)$. Note that the number of edges of a face of the dual graph corresponds to twice the multiplicity of the corresponding intersection of pseudolines in the arrangement. Therefore the zonotopal dual of an arrangement will be a rhombic tiling if and only if it is a simple arrangement. Zonotopal tilings are in one to one correspondence with arrangements of pseudolines. This, as well as existence and uniqueness of zonotopal duals, is proved in [3, Chapter 6.4].

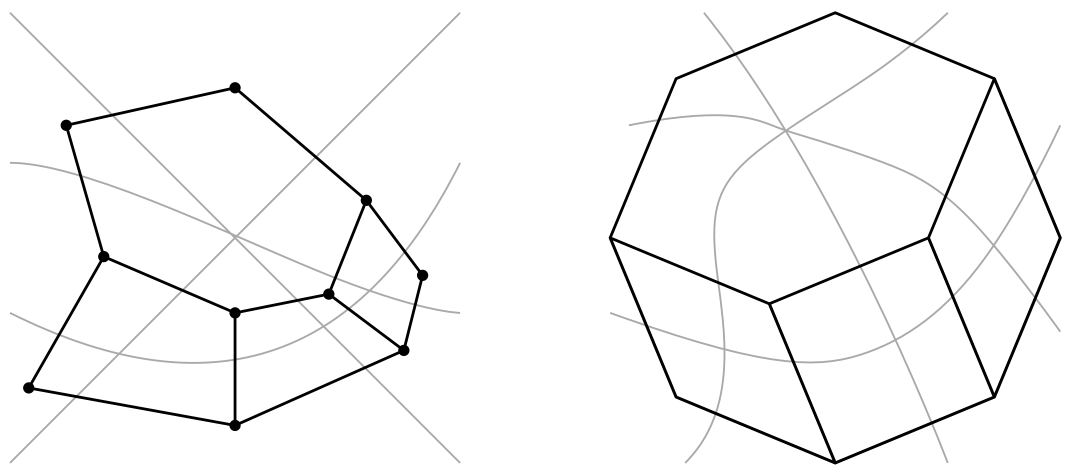

Figure 6: Left: An arrangement with its dual graph. Right: The dual graph as a zonotopal tiling.

3.2 Some Previous Lower Bounds

The first approach we are going to look at was sketched in [10, Chapter 6.2] and is the most straightforward. We start with n lines and make 3 strips of m lines each, where $m := \left\lfloor \frac{n}{3} \right\rfloor$ or $m := \left\lfloor \frac{n}{3} \right\rfloor - 1$, whichever is odd. The remaining $r = n - 3m$ lines will not be considered. Here we have $0 \leq r \leq 5$, in particular r is bounded as function of n. The strips should have slopes $1, -1$ and 0. The lines with slopes -1 and 1 form a regular grid. We choose the horizontal lines in such a way that we obtain only triple intersections. Then the horizontal lines pass through a total of $m + 2\sum_{k=1}^{\frac{m-1}{2}}(m-k) = \frac{3m^2+1}{4}$ grid vertices, compare Figure 7. We now construct simple partial arrangements by resolving the three-wise crossings locally. For every three-wise crossing we have a choice to route the horizontal line above or below the grid vertex, which gives us the lower bound $F_3(n) \geq 2^{\frac{3m^2+1}{4}} = 2^{\frac{3}{4}\frac{n^2}{9} - O(n)} = 2^{\frac{n^2}{12} - O(n)}$ on the number of options. With $c' := \frac{1}{12}$ and Lemma 2 we get the bound $c_{\sup} \geq \frac{1}{8} = 0.125$.

One way to improve on this bound is to stick with three strips of lines, but sharpen the

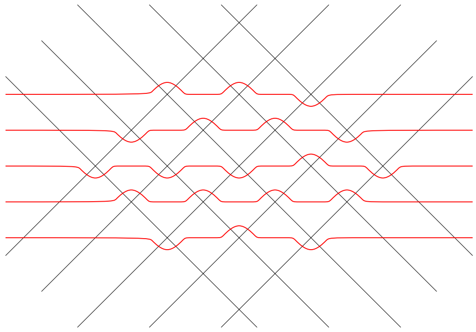

Figure 7: A possible simple partial arrangement resulting from local perturbations of the horizontal lines in a construction using three strips.

bound on $F_3(n)$. This was done by Felsner and Valtr in [4] who not only bounded, but exactly determined $F_3(n)$ using the zonotopal duals of specific partial arrangements. The zonotopal duals of simple partial arrangements that are consistent with the three strip construction are exactly the rhombic tilings of a centrally symmetric hexagon $H(m,m,m)$ with side lengths m [4, Section 4]. A more general problem, the enumeration of rhombic tilings of $H(j,i,k)$, was solved by MacMahon [9]. There are

$$PP(i,j,k) = \prod_{a=0}^{i-1}\prod_{b=0}^{j-1}\prod_{c=0}^{k-1}\frac{a+b+c+2}{a+b+c+1}$$

of these tilings. This can be approximated, yielding

$$\log_2 F_3(n) = \log_2 PP(m,m,m) \approx \log_2 e^{\left(\frac{9}{2}\ln 3 - 6\ln 2\right)m^2} = \left(\frac{9}{2}\ln 3 - 6\ln 2\right)\frac{\log_2 e}{9}n^2 - O(n).$$

Using Lemma 2 we get the lower bound $c_{\text{sup}} \geq (\frac{9}{2}\ln 3 - 6\ln 2)\frac{\log_2 e}{6} > 0.1887$, which was the best bound at the time.

Since $F_3(n)$ was not just bounded but determined exactly, this approach exhausts the potential of constructions with three strips of lines. Another strategy to improve the first approach is to keep resolving the multi-crossings locally, but consider constructions with more strips. This was done by Dumitrescu and Mandal in [1]. Here setups with 4,6,8 and 12 strips were considered, see Figure 8. A crossing of i lines in one point can be resolved in B_i non-isomorphic ways. By carefully counting the number $\lambda_i(n)$ of i-wise crossings, a bound $F_k(n) \geq \prod_{i=3}^{k} B_i^{\lambda_i(n)}$ is obtained. The best resulting lower bound comes from a twelve strip construction, which yields $c_{\text{sup}} > 0.2083$.

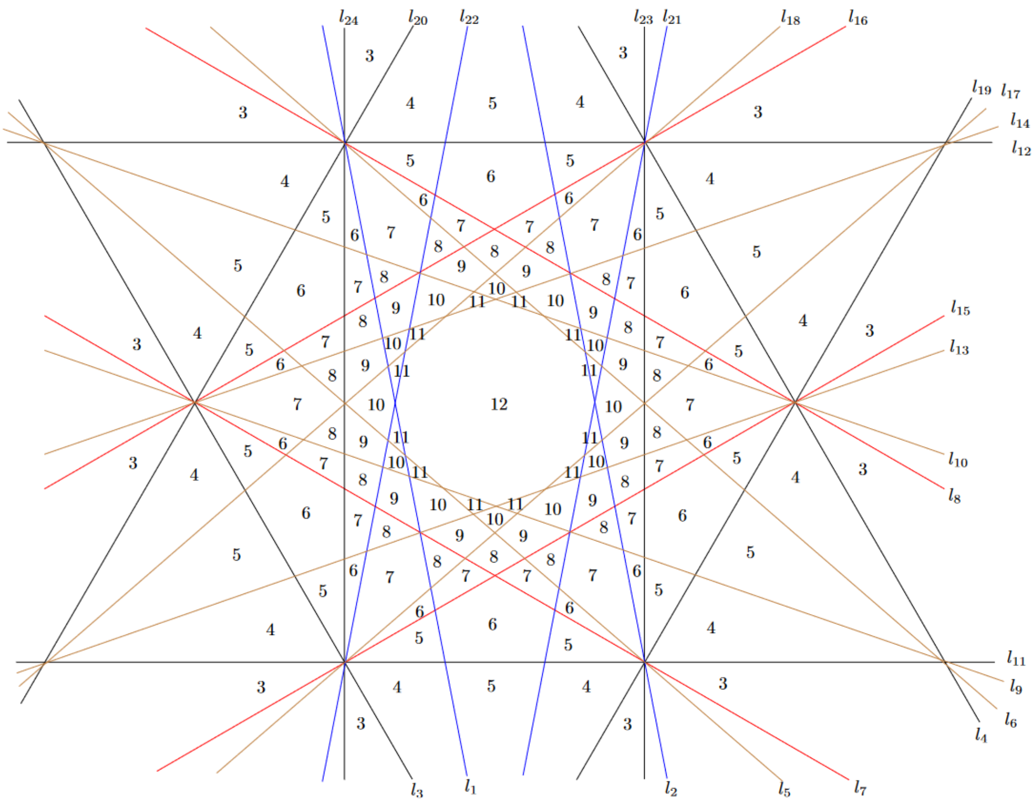

Figure 8: A construction using 12 strips. The labels ℓ_i show the outermost lines in each strip. The numbers indicate how many strips overlap in each region. This Figure is copied from [1].

4 A New Lower Bound

The general idea for the construction described below arose due to a comment by Rote to Felsner, suggesting that Dumitrescu and Mandal could have used larger patches to resolve the multi-crossings in their construction, instead of doing so locally. My task for this bachelor's thesis was to put this idea into action. This will be done in Section 4.1. In Section 4.2 we explain an algorithm that computes the number B_σ of partial arrangements consistent with any given bipermutation σ. Some constants computed using this algorithm will feature in the proof of the new bound.

4.1 Main Theorem

We begin this section by outlining the idea. We employ the same strategy that was explained in Section 2.3, this time with a construction using $k = 4$ strips. For $n \in \mathbb{N}_{>4}$ let $m := \lfloor \frac{n}{4} \rfloor$ or $m := \lfloor \frac{n}{4} \rfloor - 1$ whichever is odd. We have $r := n - 4m \leq 8$, a constant. The four strips of m lines should have slopes $0, \infty, 1$ and -1 and should be arranged as is shown in Figure 9. In particular the *primary lines* which have slopes 0 or ∞ form a grid with the grid vertices coinciding with lattice points (i, j), $i, j \in \mathbb{N}$ and the *secondary lines* of slope 1 or -1 form a grid where grid vertices lie on points of the form (i, j) or $(i \pm \frac{1}{2}, j \pm \frac{1}{2})$, $i, j \in \mathbb{Z}$.

All of the multi-crossings lie within the region $[0, m-1]^2$. Our plan is to split this region into a number $N_m \in \mathbb{N}$ of smaller patches U_i, $i \in [N_m]$, which will be considered separately. As we will see in the proof of Theorem 3, there will be only two types of patches U_i, each type of fixed size. The green regions in Figure 10 are examples of patches U_i, as we will see below. Each U_i induces a partial arrangement. It is obtained by taking the intersection of the lines with U_i and extending the resulting curves by rays starting from a shared point in the interior of U_i. Let σ_i denote the boundary bipermutation of that partial arrangement. Since the lines in the induced partial arrangement only cross within U_i, the bipermutation σ_i can be read off from the boundary of U_i. We can now resolve the multi-crossings within U_i by exchanging the line segments on U_i with any partial arrangement consistent with σ_i. This can be done independently for each U_i. Each choice yields an inequivalent partial arrangement of all of the pseudolines and we obtain the bound $F(n) := F_4(n) \geq \prod_{i=1}^{N_m} B_{\sigma_i}$.

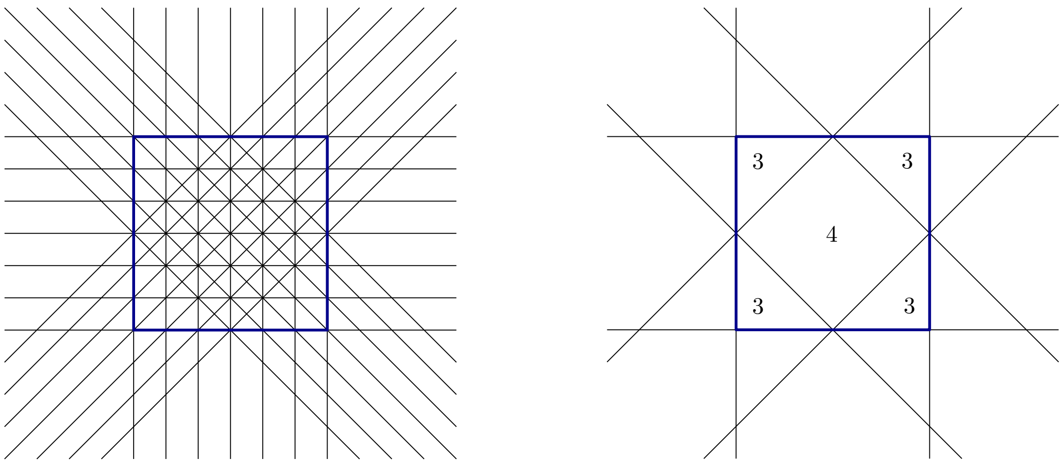

Figure 9: Left: The construction using four strips, each containing $m = 7$ lines. Right: The regions R_3, R_4, in which 3 or 4 strips overlap. The square $[0, m-1]^2$ is shown in blue. This figure is inspired by a similar one in [1].

Theorem 3. *The number B_n of non-isomorphic simple pseudoline arrangements satisfies the inequality $B_n \geq 2^{cn^2 - O(n \log n)}$ with $c > 0.2144$. In particular we have $B_n \geq 2^{cn^2}$ for all sufficiently large n.*

Proof. Consider the partial arrangement of $4m$ lines that was described above. Let R_i, $i = 3, 4$ be the region covered by exactly i strips and let a_i be the area of that region, see Figure 9. It is easy to see that $a_3 = a_4 = \frac{(m-1)^2}{2}$. We will look at R_3 and R_4 in more detail, separately.

Let R_3' be one triangular part of R_3 with area $a_3' = \frac{a_3}{4}$. We will cover it using squares $[i, i+q] \times [j, j+q]$, $i, j \in \mathbb{N}$ of size $q \in \mathbb{N}$, see the square of side length q in Figure 10. (To avoid crossings on the boundary of the squares, they will be shifted towards the right angled corner of R_3' by a small amount.) Each of these squares induces a partial arrangement. We

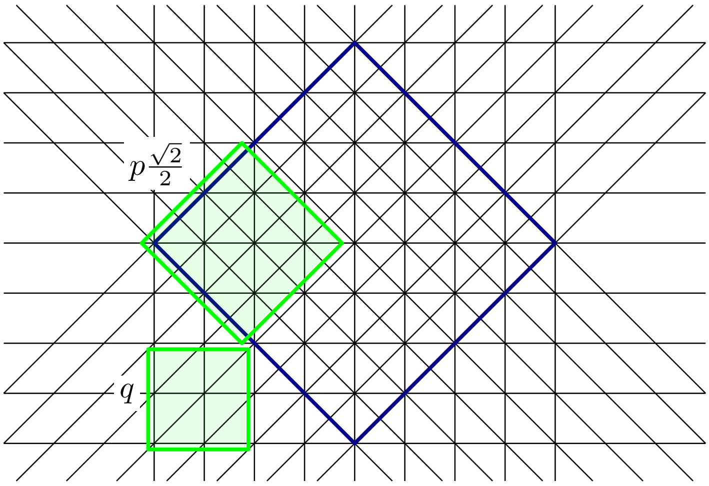

Figure 10: A four strip construction with $m = 9$ lines per strip. In green, the types of squares we use to cover R_3 and R_4. Here we chose $q = 2$ and $p = 4$.

will arbitrarily choose the top left corner of each square as its north-pole. Clearly, all the squares completely contained in R'_3 lead to isomorphic partial arrangements, as one can be transformed into the other by a translation of the plane. In particular, all the induced partial arrangements have the same bipermutation, up to relabeling of the pseudolines. Furthermore, if we cover the other components of R_3 in the same fashion, the resulting partial arrangements will also be isomorphic, this time by rotation as well as translation. As a result, all the bipermutations of squares contained in R_3 are the same up to cyclic shift and relabeling. In particular, each has the same number $B_{\sigma_3(q)}$ of consistent partial arrangements, where $\sigma_3(q)$ is one representative of the set of bipermutations induced by squares completely contained in R_3.

We will now count the number $\lambda_3(n)$ of these squares. Let s denote the area of R'_3 not covered by squares that are completely contained in it, see the dark red area in Figure 11. From the figure it is immediately clear, that s is less than $2(\frac{m-1}{2} + q)q \in O(n)$. Therefore there are $\frac{a'_3 - s}{q^2} = \frac{a'_3}{q^2} - O(n) = \frac{1}{2^7 q^2} n^2 - O(n)$ of these squares in R'_3. Since there are four such regions the total number of squares is $\lambda_3(n) := \frac{1}{2^5 q^2} n^2 - O(n)$.

Now consider R_4. We will cover it using squares that align with the grid of secondary lines, see Figure 10. Each square should cover p^2 secondary grid vertices each, where p is even. The side lengths of the squares will be $p\frac{\sqrt{2}}{2}$. We will choose the left most corner of each square as its north-pole. Again the induced bipermutations corresponding to each rectangle only differ by relabeling, because one induced partial arrangement can be translated onto the other. Let $B_{\sigma_4(p)}$ be the number of partial arrangements consistent with these bipermutations. The number of rectangles completely contained in R_4 is $\lambda_4(n) := \lfloor \frac{m-1}{p} \rfloor^2 = \frac{1}{2^4 p^2} n^2 - O(n)$. (Alternatively we can use an argument similar to the one we used for R'_3, also yielding $\lambda_4(n) = \frac{a_4}{\left(p\frac{\sqrt{2}}{2}\right)^2} - O(n) = \frac{1}{2^4 p^2} n^2 - O(n)$.)

We can replace each part of the arrangement of straight lines covered by one of the squares by any partial arrangement consistent with the corresponding bipermutation. The result will always be a legal arrangement, because Proposition 2 ensures, that the same pseudolines cross in the modified region. Furthermore, each choice produces non-isomorphic arrangements by Corollary 2.1. All in all we have

$$\log_2 F(n) \geq \log_2 \left(B_{\sigma_3(q)}^{\lambda_3(n)} \cdot B_{\sigma_4(p)}^{\lambda_4(n)} \right) = n^2 \left(\frac{\log_2(B_{\sigma_3(q)})}{2^5 q^2} + \frac{\log_2(B_{\sigma_4(p)})}{2^4 p^2} \right) - O(n). \quad (3)$$

Inequality (3) holds for any q and even p, but to produce a bound we also need to determine $B_{\sigma_3(q)}$ and $B_{\sigma_4(p)}$ for those q, p we want to use. We have done that computationally, as will be explained in Section 4.2. Here we will just use their computed values for $q = 10$ and $p = 8$. They are $B_{\sigma_3(q)} \approx 1.96 \cdot 10^{39}$ and $B_{\sigma_4(p)} \approx 1.02 \cdot 10^{37}$. The exact values of $B_{\sigma_3(q)}$ and $B_{\sigma_4(p)}$ for these and some other choices of p, q can be found in Tables 1 and 2. Inserting the values into inequality (3) yields the bound $\log_2 F(n) \geq 0.1608n^2 - O(n)$. Finally we apply Lemma 2, which yields the bound $B_n \geq 2^{cn^2 - O(n \log n)}$ for some $c > 0.2144$. $\square$

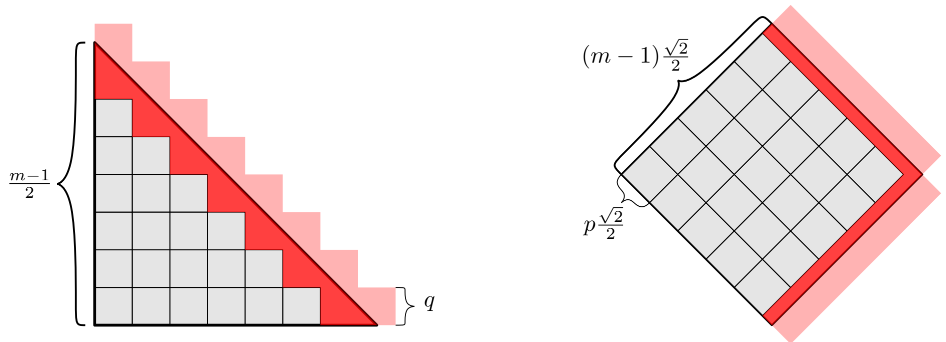

Figure 11: Left: A partial covering of R_3' with (grey) squares of side length q. Right: A partial covering of R_4 using squares of side length $p\frac{\sqrt{2}}{2}$.

4.2 Algorithm to Calculate B_σ

Of course the above approach would not have worked without knowing B_σ for a few specific σ. In this section we explain Algorithm 1, which calculates B_σ for any bipermutation σ. It employs a divide-and-conquer strategy by splitting the problem along one pseudoline (see Figure 12) and solving the resulting subproblems recursively before recombining the results arithmetically.

A simpler approach, inserting the pseudolines into the partial arrangement one by one and thereby explicitly constructing all the consistent partial arrangements, only terminates in reasonable time for regions in R_4 of side length at most 4 to 6. In a conversation Scheucher suggested exploiting the grid structure in R_4 to go from 4×4 regions to 4×8 and potentially even larger regions. (The proof of Theorem 3 also works with rectangular regions instead of squares.) To do this, one would split the 4×8 region into two 4×4 squares and construct all possible orders in which the pseudolines could exit the first square and enter second one. Each such possibility yields boundary bipermutations for the two squares. For each order, the number of consistent partial arrangements in the two squares would be calculated and multiplied. Since the number of ways the pseudolines can cross from one square into the other is much lower than the number of consistent partial arrangements in each square, this method finishes its calculation in close to the square root of the time it would take to construct all the consistent partial arrangements explicitly. The following algorithm takes this approach a step further. It takes any bipermutation as input and decides on one pseudoline to split the problem along (similarly to how we were going to split the 4×8 region into two 4×4 regions). The numbers of consistent partial arrangements of the two sub regions will be calculated recursively, which is only possible with this more general algorithm.

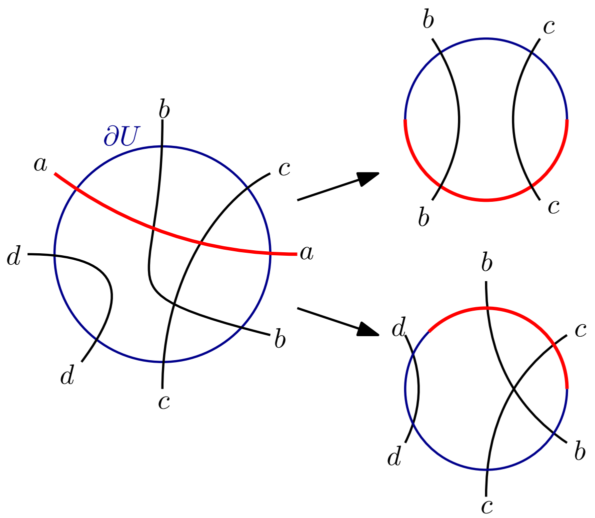

Figure 12: The given partial arrangement of $n = 4$ pseudolines is split along the pseudoline ℓ_a. This yields two partial arrangements of at most $n-1$ pseudolines. If ℓ_b and ℓ_c were crossing on the other side of ℓ_a, two different partial arrangements would be created.

Splitting the Problem

Let σ be the bipermutation, of which we want to calculate the number B_σ. All operations in the final algorithm will be on the level of bipermutations, not partial arrangements. However it is very useful to conceptualize the steps in terms of partial arrangements. Let $\mathcal{A}$ be any partial arrangement consistent with σ and let U be a disc containing all of the crossings in $\mathcal{A}$. We will first explain, what it means to split $\mathcal{A}$ along a pseudoline ℓ_a, see Figure 12. Let U_L, U_R be the two connected components of $U \setminus \ell_a$ and let $\mathcal{A}_L$, $\mathcal{A}_R$ the two partial arrangements that $\mathcal{A} \setminus \ell$ induces on U_L and U_R. Finally let $\sigma_L(\mathcal{A})$, $\sigma_R(\mathcal{A})$ be the bipermutations of these induced arrangements. The result of splitting $\mathcal{A}$ along ℓ are the two partial arrangements $\mathcal{A}_L$, $\mathcal{A}_R$.

Splitting the problem of calculating B_σ along the element a means constructing not just the pair $(\sigma_L(\mathcal{A}), \sigma_R(\mathcal{A}))$, but constructing the set $\{(\sigma_L(\mathcal{A}'), \sigma_R(\mathcal{A}')) \mid \mathcal{A}' \text{ is consistent with } \sigma\}$ of all such pairs resulting from partial arrangements consistent with σ.

The Recursion Step by Step

The first step is to choose an element a along which to split the problem. We want both of the subproblems to be as small as possible. There are different ways to specify this condition. Given a pseudoline ℓ the partial arrangement $\mathcal{A}$ splits into two partial arrangements $\mathcal{A}_1$, $\mathcal{A}_2$ of k_1, k_2 pseudolines, respectively, where $k_1 \geq k_2$. We will prefer ℓ to another pseudoline ℓ' with corresponding data (k_1', k_2'), where $k_1' \geq k_2'$ if

$$(k_1, k_2) \leq (k_1', k_2') :\Longleftrightarrow k_1 < k_1' \vee (k_1 = k_1' \wedge k_2 \leq k_2').$$

This amounts to focusing on the size of the larger of the two components and only considering the smaller ones when comparing two pseudolines that produce larger components of the same size. Of course our algorithm does not depend on the choice of $\mathcal{A}$. The values (k_1, k_2) can be computed from σ directly as follows. Each element $a \in [n]$ splits σ into two sections $(b_1, \dots b_i)$, $(c_1, \dots c_j)$ of lengths i, $j \in [2n]$ such that $\sigma = (a, b_1, \dots b_i, a, c_1, \dots c_j)$ after an appropriate shift operation. Let α be the number of elements crossing a in σ. That is the number of elements that appear both in $(b_1, \dots b_i)$ and in $(c_1, \dots c_j)$. Now the sizes $\{k_1, k_2\}$ of the subproblems produced by a split along the pseudoline corresponding to a in any consistent partial arrangement are $\{\frac{i+\alpha}{2}, \frac{j+\alpha}{2}\}$, compare Figure 12.

Next we need to construct all possible bipermutations that can result from the split. Let $\eta_a(\mathcal{A})$ record the order in which the other pseudolines in $\mathcal{A}$ cross the pseudoline ℓ corresponding to a. The bipermutation of the two subproblems produced by a are

$$\sigma_L(\mathcal{A}) = (b_1, \dots b_i, \eta_a(\mathcal{A})),$$

$$\sigma_R(\mathcal{A}) = (\tilde{\eta}_a(\mathcal{A}), c_1, \dots c_j),$$

where $\tilde{\eta}_a(\mathcal{A})$ is $\eta_a(\mathcal{A})$ in reversed order, compare Figure 12. We observe that there are restrictions (applying to all consistent partial arrangements) on the order in which the pseudolines crossing ℓ can do so. Let p_1, p_2 be the two intersections of ℓ and ∂U. Let ℓ_i, ℓ_j be two pseudolines crossing ℓ. If ℓ_i, ℓ_j themselves do not cross and ℓ_i comes before ℓ_j when walking along ∂U from p_1 to p_2, then ℓ_i also must come before ℓ_j when walking along ℓ. This gives rise to a partial order P on the set of pseudolines crossing ℓ, see Figure 13. All orderings in the image of η_a have to respect this partial order. On the other hand, every such ordering is realized by at least one consistent partial arrangement, which follows quickly from Propositions 1 and 2. Therefore the image of η_a is exactly the set T of linear extensions of P. With this we can construct the bipermutations $\sigma_L(t), \sigma_R(t)$ of the left and right sides for each $t \in T$. Finally, we calculate $B_{\sigma_L(t)}, B_{\sigma_R(t)}$ for all $t \in T$ recursively and return the recombined results $B_\sigma = \sum_{t \in T} B_{\sigma_L(t)} B_{\sigma_R(t)}$.

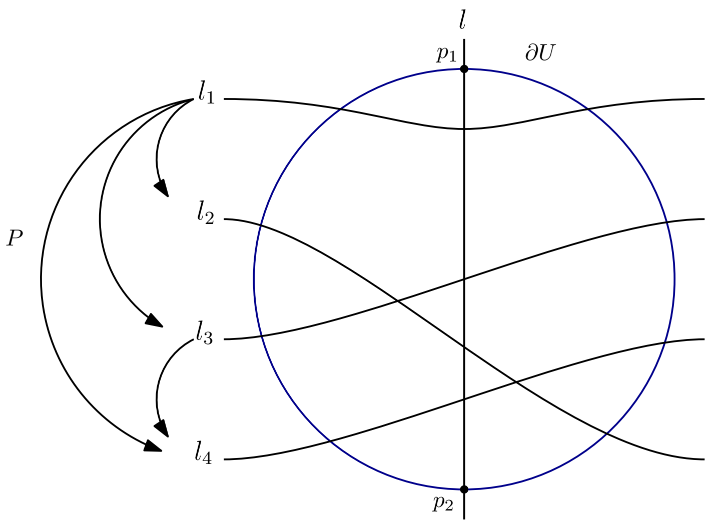

Figure 13: The partial arrangement will be split along ℓ. The order in which the pseudolines $\ell_1, \ell_2, \ell_3, \ell_4$ cross ℓ has to be compatible with the partial order P. For example: ℓ_1 has to come first in any consistent partial arrangement, but whether ℓ_2 comes second, third or fourth depends on the partial arrangement.

Storing Intermediate Results

This algorithm can be improved by storing the B_σ after they are first calculated instead of recalculating them whenever they appear. This was also suggested by Scheucher. Particularly bipermutations on a smaller number of elements tend to appear very often in the recursion. For two bipermutations σ, σ' that can be transformed into each other by a combination of shift, reflection and relabeling of the pseudolines we have $B_\sigma = B_{\sigma'}$. Therefore it is more efficient (at least in terms of the number of times our main function is called) to get a unique representative of each such class of bipermutations and only save the result once. As a representative we will choose the lexicographically least member of the class. Note, that each class contains at least $n!$ members, since any permutation of the elements will result in a distinct bipermutation in the same class. Now we can use a trick, so we don't have to iterate over all of those members.

We endow the set $[2n]$ with the metric defined by

$$d(i,j) := \min(|j-i|,\ 2n - |j-i|), \quad i,j \in [2n].$$

It is the natural metric on the set of vertices of a cycle graph of length $2n$. Given a bipermutation $\sigma \in S_n^{(2)}$ we define a function $d_\sigma : [n] \to \mathbb{N}$ by

$$d_\sigma(a) := \mathrm{diam}(\sigma^{-1}(a)), \quad a \in [n],$$

where the diameter is with respect to the metric d on $[2n]$. More explicitly, if $\sigma(i) = \sigma(j) = a$, then $d_\sigma(a) = d(i,j)$. We will call d_σ the *diameter function* and (by a slight abuse of notation) its value $d_\sigma(a)$ the *diameter* of a.

Let a be an element for which d_σ is minimal and let $\{i,j\} := \sigma^{-1}(a)$, $i < j$. We can assume $i = 1$ and $d_\sigma(a) = j-1$ otherwise shift appropriately. If we start relabeling σ greedily with respect to the lexicographical ordering, we acquire a new bipermutation $\sigma' = (1,2,3,\dots d_\sigma(a), 1, *\,*\,*)$. The section $(1,\dots d_\sigma(a), 1)$ cannot be improved upon, since that would require an element of smaller diameter than a. Switching i and j using a reflection and greedily labeling the resulting bipermutation leads to another candidate for the representative. We also could have started with a different minimal element. That results in at most $2n$ candidates to check, which is much better than $n!$. In the worst case it takes $2n$ comparisons between bipermutations, so $O(n^2)$ operations in total. In practice this worst case is very rare. Empirically there are about 1.4 minimal elements on average, when working with the partial arrangements we needed for our proof, which had up to 30 pseudolines.

```
function CALCULATE_B(σ)
    NumberOfLines ← |σ|/2
    if NumberOfLines ≤ 2 then return 1
    if ALREADY_CALCULATED(σ) then return STORED_VALUE(σ)

    a ← SELECT_CURVE_LABEL(σ)
    P ← GET_PARTIAL_ORDER_ON_CROSSING_CURVES(a, σ)
    T ← COMPUTE_ALL_LINEAR_EXTENSIONS(P)

    return ∑_{t∈T} CALCULATE_B(σ_L(t)) · CALCULATE_B(σ_R(t))
```

Algorithm 1: The algorithm described in the above section. A C++ implementation of the algorithm can be found in this github repository: https://github.com/corteskuehnast/PLA

5 Conclusion and Outlook

We have used an approach based on partial arrangements to bound from below the number of pseudoline arrangements, yielding the new bound $B_n \geq 2^{0.2144n^2}$ for all sufficiently large n. As part of our approach we studied partial arrangements in detail, proving some general properties of partial arrangements and bipermutations. In addition we provided an algorithm to calculate the number of partial arrangements consistent with a given bipermutation, which was also central to the approach.

Some rough preliminary estimates suggest that applying this method to constructions with more slopes (such as those used in [1]) would lead to more significant improvements to the lower bound, but would also require a lot more attention to detail, since the regions in those constructions are not as regular.

6 References

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A Tables of Values of B_σ

The following tables contain some data that was needed to prove Theorem 3. In the proof we derived the lower bound

$$\log_2 F_4(n) \geq n^2 \left(\frac{\log_2(B_{\sigma_3(q)})}{2^5 q^2} + \frac{\log_2(B_{\sigma_4(p)})}{2^4 p^2} \right) - O(n)$$

on the number of possible interactions between the strips of pseudolines. This can be translated into the bound

$$c_{\sup} \geq \frac{4}{3} \left(\frac{\log_2(B_{\sigma_3(q)})}{2^5 q^2} + \frac{\log_2(B_{\sigma_4(p)})}{2^4 p^2} \right) = c_4 + c_3$$

with

$$c_4 := c_4(\sigma_4, p) := \frac{4}{3} \frac{\log_2(B_{\sigma_4(p)})}{2^4 p^2},$$

$$c_3 := c_3(\sigma_3, q) := \frac{4}{3} \frac{\log_2(B_{\sigma_3(q)})}{2^5 q^2}$$

using Lemma 2. The bounds c_4, c_3 on $c_{\sup}$ can be interpreted as the bounds one would get by focusing solely on the region R_4, R_3 and only using that it is possible to fill in the other region.

We display these bounds, as well as the corresponding $B_{\sigma_4(p)}$, $B_{\sigma_3(q)}$, for some choices of p, q in the tables below. Since the bipermutations σ_4, σ_4 themselves are too long to comfortably fit the page, we will refer to them using only the sidelengths of the regions they are induced by, see Figure 10. Note, that the sidelengths in Table 1 are with respect to the grid that is formed by the secondary lines. To get the geometric length just multiply by $\frac{\sqrt{2}}{2}$. We also show the time spent calculating the B_σ using the algorithm described in Section 4.2. More specifically we used the C++ implementation of the algorithm that can be found in this github repository https://github.com/corteskuehnast/PLA. There the σ_4 are also present. All calculations were done on the compute cluster of the TU Berlin.

R_4

side length p	$B_{\sigma_4(p)}$	c_4	time
2	82	0.1324	0s
4	346814370	0.1477	0s
6	1689545857396764814888	0.1555	275s
8	10233480626615962155895931163981261674	0.1600	$\sim$18h[1]

R_4 (rectangular regions)

side lengths	$B_{\sigma_4(p)}$	c_4	time
2×3	890	0.1360	0s
3×4	1810562	0.1443	0s
4×5	67355906900	0.1498	0s
4×7	2559684720337354	0.1523	2s
4×9	97487823328918489282	0.1537	7s
5×6	69013961584897708	0.1553	8s
6×7	611720737105383837357810	0.1567	138s
7×8	141024673497994643753371170615708	0.1589	$\sim$3h[2]

Table 1: A table of $B_{\sigma_4(p)}$ for some σ_4 of the type used in the proof of Theorem 3. The entry used in the proof is from the $p = 8$ row. To provide more context we also included some data resulting from rectangular regions, instead of squares. With the side lengths $p_1 \times p_2$ we derive the lower bound $c_{\text{sup}} \geq c_4 := \frac{4}{3}\frac{\log_2(B_{\sigma_4(p_1,p_2)})}{2^4 p_1 \cdot p_2}$ analogously to the square case.

R_3

side length q	$B_{\sigma_3(q)}$	c_3	time
2	20	0.0450	0
3	1320	0.0479	0
4	592116	0.0499	0
5	1822326492	0.0512	0
6	38646198270218	0.0522	1s
7	5660877091974830660	0.0529	5s
8	5735809610253552915456670	0.0535	237s
9	40240394566420231438640750723072	0.0540	632s
10	1956055471674766249002559523437101670400	0.0543	$\sim$ 6h[2]

Table 2: A table of $B_{\sigma_3(q)}$ for some σ_3 of the type used in the proof of Theorem 3. The entry used in the proof is from the $q = 10$ row.

[1]This problem was split into 64 sub problems that were calculated separately. Calculating it in one go would take longer, but likely not 64 times longer, since memory was not shared between the sub processes.

[2]This problem was split into 100 sub problems.

Zusammenfassung

In dieser Arbeit wird die neue untere Schranke $B_n \geq 2^{cn^2 - O(n \log(n))}$ für ein $c > 0.2144$ der Anzahl B_n an Äquivalenzklassen von markierten Arrangements gezeigt. Bisher bekannt war $c > 0.2083$. Der Beweis beruht auf der Einteilung der Ebene in Flicken. Mittels Computerberechnungen wird die Anzahl an Möglichkeiten berechnet, wie sich die Pseudogeraden in einem Flicken verhalten können. Dabei ist das Verhalten am Rand der Flicken vorgegeben. Das Zusammensetzen dieser Flicken erzeugt dann die benötigte Anzahl an Arrangements.

Dazu werden partielle Arrangements und ihre Bipermutationen definiert, die die Idee der Flicken präzisieren. Einige grundlegende Eigenschaften von partiellen Arrangements werden gezeigt, die später in den Beweis der neuen Schranke einfließen. Schließlich wird ein Algorithmus zur Berechnung der Anzahl an Möglichkeiten in einem Flicken angegeben.